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Asset flow and momentum: deterministic and stochastic equations

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We use basic conservation and microeconomic identities to derive a nonlinear first-order ordinary differential equation for a market system with a prescribed number of shares and cash supply (including additions in time).

The equation incorporates the ideas of the finiteness of assets and preference that is influenced by price momentum and discount from fundamental value. The concept of a 'liquidity value', defined as the total cash in the system divided by the number of shares, emerges as a key price along with the fundamental value. In the absence of a clear focus on fundamentals, the price evolves into the liquidity value. This is consistent with the belief of some market analysts who feel that liquidity, or a large sum of cash available for investment, is a primary factor in moving asset prices higher.

These equations can also be derived from the system of equations used in previous work by considering a closed system and taking the limit of short time-scale in the preference or transition function as well as some linearization.

Finally, the full system of equations is generalized to include randomness. The resulting stochastic system is studied numerically. In particular, when the deterministic equations are complemented with randomness, the solutions generate a range of stochastic patterns, such as the head and shoulders with certain characteristics in common.

Keywords: liquidity; momentum; asset flow; price dynamics; stochastic equations

1. Introduction

Financial market analysts often assert that valuation is only one of several factors that determine the price of an asset and its time evolution. Among the other key factors are price momentum and the concept of the finiteness of assets. The latter means that when most of the possible cash that can be used to purchase an asset is already invested, the potential for higher prices is limited even if other factors are positive. In the stock exchanges, one manifestation of the available cash is the average percentage of all mutual funds that is in cash. Also, a drop in interest rates significantly raises the pool of cash for stocks as investors become dissatisfied with prevailing yields and turn to stocks.

In the first part of this paper, we consider a single asset market in which the number of shares and amount of cash is constant in time. We use simple conservation laws, along with basic adjustment to excess demand and a transition or preference function that depends upon the discount from fundamental value and the price derivative. The resulting ordinary differential equation for price as a function of time is first order and nonlinear. While the fundamental value $P_{\mathrm{a}}(t)$ is explicitly part of the equation, an additional price emerges as a natural price unit. This is the 'liquidity value' defined as the total cash divided by the number of shares. The analysis of § 2 shows that in the absence of clear emphasis on a fundamental value, the equilibrium value will be close to the liquidity value. Of course, emphasis on the price derivative will generally result in larger bubbles and subsequent crashes.

The ordinary differential equation is easily generalized to include the additional flow of cash and shares that often play a role depending upon the market situation or experimental design.

Experimental asset markets have become an important tool in understanding basic market phenomena, particularly since the experiments can be repeated and varied to test specific ideas. In particular, the laboratory asset markets devised by Smith *et al*. (1988), Williams & Smith (1984), Davis & Holt (1993), Plott (1986) and Smith (1982) established a number of trading periods in order to examine the time evolution of the trading prices and volume. A standard 'bubbles' experiment involves nine participants who are given some distribution of cash and shares of an asset or security which will pay a dividend, with expected dividend of 24 cents, at the end of each of the 15 periods. Thus the realistic or 'fundamental' value of the asset is clearly \$3.60 at the outset of the experiment and declines stepwise by \$0.24 each period until it becomes worthless after the fifteenth period. Classical theories of economics or finance, such as the rational expectations, would predict a time evolution of the trading price that is similar to this fundamental value with some fluctuations due to randomness of trading.

In the experiments, however, one usually observes an initial period trading price that is well below the realistic value of \$3.60, followed by rising prices that overshoot the fundamental value in the intermediate periods, creating a characteristic 'bubble' and a dramatic 'crash' of prices near the end of the experiment.

The experiments with no uncertainty about the expected dividends (Porter & Smith 1994), in particular, draw attention to the idea that the actions and strategies of other traders can provide the only element of uncertainty to participants.

A system of differential equations, discussed in § 3, has been used to study a variety of issues including (a) the qualitative and quantitative price behaviour in an asset market; (b) the price dynamics of a major market crash; (c) phenomena such as persistent undervaluations in markets; (d) the origin of a speculative bubble; and (e) the origin of price patterns, known as technical analysis or charting.

These equations have been derived with the assumption that the traders are a small part of the investment pool, so there is no conservation of cash, for example. On the other hand, the single equation derived below stipulates a closed system so that the total shares and cash are conserved, except for specified additions and withdrawals. This enables an examination of the issue of liquidity and its implications for asset markets. A system of equations similar to the original is also derived under this assumption in § 3.

Finally, in § 4, we examine the effects of randomness on the patterns generated by a set of fundamental value functions which exhibit a steady change or turning point.

2. A single first-order ODE for a closed system

We derive a nonlinear first-order ordinary differential equation for a market to which no new shares or money is added. Using some basic identities, one can derive a rather simple equation for the price of the asset. This equation can also be regarded as an approximation of the full system defined in § 3. The equation is easily generalized, as shown below, to the case in which more shares or dollars are added to the market.

We consider a closed market containing N shares and a total of M dollars distributed arbitrarily among participants at the outset.

Let the price of the single asset be denoted again by $P(t)$ and define the constant, the liquidity value, $L := M/N$, which also has units of dollars. Let B denote the fraction of total funds in the asset.

The equation can be derived from the following assumptions.

(A) No additional shares or cash are added to the system.

(B) The demand D is the total cash supply times the transition rate k, or the probability that one unit of cash will be used to place an order, and likewise for the supply S. One then has

$$D = k(1-B), \qquad S = (1-k)B, \qquad \frac{D}{S} = \frac{k}{1-k}\frac{1-B}{B}. \tag{2.1}$$

(C) The transition rate k is a weighted sum of the current derivative and the valuation discount, i.e.

$$k := \tfrac{1}{2}(1+\zeta), \qquad \zeta := \zeta_1 + \zeta_2, \qquad \zeta_1 = \frac{q_1\tau_0}{P}\frac{\mathrm{d}P}{\mathrm{d}t}, \qquad \zeta_2 = q_2\left(1 - \frac{P(t)}{P_{\mathrm{a}}(t)}\right). \tag{2.2}$$

(D) The relative price changes linearly with excess demand, i.e.

$$\frac{\tau_0}{P}\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{D}{S} - 1, \tag{2.3}$$

for a time-scale τ_0. This is a limiting form (i.e. derivative instead of difference) of a standard microeconomic assumption (see, for example, Henderson & Quandt 1980, p. 162).

Note that we only need the third equation of (2.1), and only $k/(1-k)$ enters into this equation. Note that in making Assumption C we have linearized the transition rate k which should take values within [0,1] in order to ensure that the probabilities are non-negative. In other words, $k := (1+\zeta)/2$ is an approximation to $k := 1/2 + (\tanh\zeta)/2$ or a logistic equivalent. In any regime of practical interest the approximation will be very close to the nonlinear definition and k will be within [0,1]. In extreme circumstances one can easily rewrite the equations with the nonlinear k.

Immediate consequences of assumption A are the identities

$$B = \frac{NP}{NP+M}, \qquad 1-B = \frac{M}{NP+M}, \tag{2.4}$$

$$\frac{B}{1-B} = \frac{N}{M}P = \frac{P}{L}, \tag{2.5}$$

so that $B^{-1}(1-B)P = L$ is time invariant. Although we do not need to use it in the derivation, equation (2.5) leads to the derivative identity

$$\frac{\mathrm{d}B}{\mathrm{d}t} = B(1-B)\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}t}. \tag{2.6}$$

Note that there is no need for a time-scale in this equation.

Assumption C leads to the relation

$$\frac{k}{1-k} = \frac{1+\zeta}{1-\zeta} = 1 + 2(\zeta_1+\zeta_2) = 1 + 2\frac{q_1\tau_0}{P}\frac{\mathrm{d}P}{\mathrm{d}t} + 2q_2\left(1 - \frac{P(t)}{P_{\mathrm{a}}(t)}\right). \tag{2.7}$$

Using (2.5) in the price equation (2.3) results in

$$\frac{\tau_0}{P}\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{k}{(1-k)}\frac{L}{P} - 1, \tag{2.8}$$

and substituting for $k/(1-k)$ using (2.7) leads to the equation

$$\frac{\tau_0}{P}\frac{\mathrm{d}P}{\mathrm{d}t} = \left[1 + 2\frac{q_1\tau_0}{P}\frac{\mathrm{d}P}{\mathrm{d}t} + 2q_2\left(1 - \frac{P(t)}{P_{\mathrm{a}}(t)}\right)\right]\frac{L}{P} - 1. \tag{2.9}$$

We note that the liquidity value L, which represents the nominal value of all money in the system divided by the total number of shares, is a fundamental scale for price. With this in mind, we rewrite equation (2.9) as

$$\tau_0\left(1 - Q_1\frac{L}{P}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{P}{L}\right) + \left(1 + Q_2\frac{L}{P_{\mathrm{a}}}\right)\frac{P}{L} = 1 + Q_2, \tag{2.10}$$

with $Q_1 := 2q_1$ and $Q_2 := 2q_2$. If we use these natural units of price and time by defining

$$\boldsymbol{P} := \frac{P}{L}, \qquad \boldsymbol{P}_{\mathrm{a}} := \frac{P_{\mathrm{a}}}{L}, \qquad \tau := \frac{t}{\tau_0},$$

then one has simply

$$\left(1 - \frac{Q_1}{\boldsymbol{P}}\right)\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}\tau} + \left(1 + \frac{Q_2}{\boldsymbol{P}_{\mathrm{a}}}\right)\boldsymbol{P} = 1 + Q_2, \tag{2.10$'$}$$

revealing a symmetry between the price and the derivative.

Some basic features of this equation are apparent upon examining equilibrium features. Setting the derivative equal to zero, and writing $\boldsymbol{P}_{\mathrm{a}}$ for $\boldsymbol{P}_{\mathrm{a}}(\infty)$, one has from (2.10$'$) the equation for the equilibrium price $\boldsymbol{P}_{\mathrm{eq}}$,

$$\left(\frac{1}{Q_2} + \frac{1}{\boldsymbol{P}_{\mathrm{a}}}\right)\boldsymbol{P}_{\mathrm{eq}} = 1 + \frac{1}{Q_2}. \tag{2.11}$$

(To interpret this, we recall that $\boldsymbol{P}_{\mathrm{a}} := P_{\mathrm{a}}/L$ is the ratio of fundamental value to liquidity value.) If the weighting of the influence of Q_2 is large (that is, $Q_2 \gg \max(1, \boldsymbol{P}_{\mathrm{a}})$, then $\boldsymbol{P}_{\mathrm{eq}} \sim \boldsymbol{P}_{\mathrm{a}}$, so that the fundamental value is attained. However, if $Q_2 \ll \min(1, \boldsymbol{P}_{\mathrm{a}})$, one has $\boldsymbol{P}_{\mathrm{eq}} \sim 1$, so that

$$P \sim L, \tag{2.12}$$

which means that the liquidity value (total dollars divided by total number of shares) is attained as an equilibrium value. *In the absence of clear information and attention to value, the price tends to gravitate to a natural value determined by the ratio of total cash to total quantity of asset.* At a very simple level this is similar to the exchange between a person who has a currency that can only be used for one commodity that is entirely owned by another person.

Solving (2.11) for Q_2 yields (in the original units)

$$Q_2 = \frac{1 - P_{\mathrm{eq}}/L}{P_{\mathrm{eq}}/P_{\mathrm{a}} - 1}. \tag{2.13}$$

This means that Q_2 is a factor that interpolates the distance from the equilibrium value P_{eq} to the fundamental value P_{a} and the liquidity value L.

In particular, for $Q_2 > 0$ (meaning one is more inclined to buy when the asset is at a discount), the equilibrium price P_{eq} is always between the fundamental value P_{a} and the liquidity value L, regardless of which of these is greater. Consequently, in a closed system in which $L \neq P_{\mathrm{a}}$, there is a competition between the two prices P_{a} and L at or near equilibrium.

Within the experimental setting, equation (2.13) gives us an important tool for evaluating Q_2, since L and P_{a} are known from the definition of the experiment while P_{eq} is known at the conclusion of the experiment. It also is an equation that is relatively easy to verify in the laboratory by varying P_{a} and L in experiments with similar populations (characterized by similar values of Q_2).

For constant $\boldsymbol{P}_{\mathrm{a}}$, equation (2.10′) has an exact solution that can be obtained by partial fractions and separation of variables. The solution is given by

$$\boldsymbol{P}^{-Q_1/\boldsymbol{P}_{\mathrm{eq}}} |\boldsymbol{P} - \boldsymbol{P}_{\mathrm{eq}}|^{1-Q_1/\boldsymbol{P}_{\mathrm{eq}}} = C\mathrm{e}^{-s\tau}, \tag{2.14}$$

where one has from (2.11)

$$\boldsymbol{P}_{\mathrm{eq}} := \frac{1 + Q_2}{1 + Q_2/\boldsymbol{P}_{\mathrm{a}}},$$

$s := 1 + Q_2/\boldsymbol{P}_{\mathrm{a}}$ and C is a generic constant from the integration.

For $Q_1 = 0$ one simply has

$$\boldsymbol{P}(\tau) - \boldsymbol{P}_{\mathrm{eq}} = (\boldsymbol{P}(0) - \boldsymbol{P}_{\mathrm{eq}})\mathrm{e}^{-s\tau}, \tag{2.15}$$

so that the price converges to the equilibrium value with a relaxation time s. Note that the rate s is at least 1, and as s increases so does Q_2, and equilibrium is approached more rapidly. Recalling that $\boldsymbol{P} := P/L$ and $\boldsymbol{P}_{\mathrm{eq}} := P_{\mathrm{eq}}/L$, the equilibrium value P_{eq} moves asymptotically close to P_{a} as Q_2 increases, while it evolves toward L as Q_2 vanishes.

Remark 2.1. (Market system with additional cash or shares.) The equation above is easily generalized to the situation in which additional cash or shares are added to the market in an arbitrary manner. Suppose the number of shares $N(t)$ and the total cash in the system $M(t)$ are allowed to vary in time with $L(t) := M(t)/N(t)$. Then the relation between B and P remains unchanged (though the derivatives no longer satisfy (2.6), and one obtains the equation

$$\tau_0 \left[1 - \frac{Q_1}{P} L(t)\right] \frac{\mathrm{d}P}{\mathrm{d}t} + \left[1 + \frac{Q_2}{P_{\mathrm{a}}(t)} L(t)\right] P = (1 + Q_2)L(t), \tag{2.16}$$

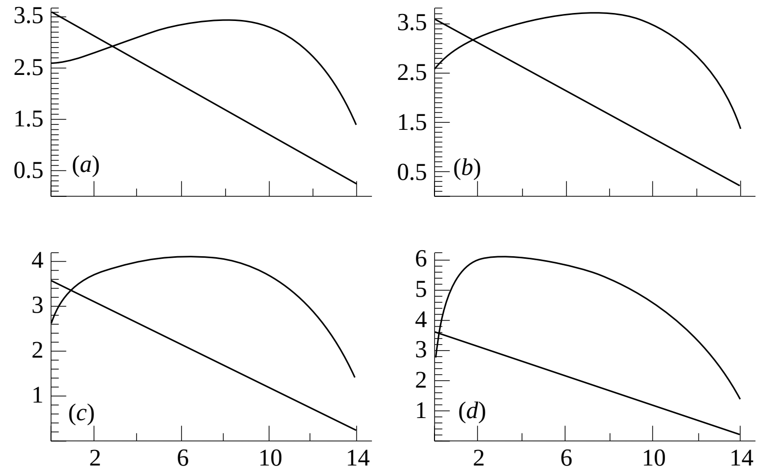

Figure 1.

which can also be placed in terms of the liquidity value $L(t)$, which is now variable in time.

An application is to the bubbles experiments (Porter & Smith 1994) in which $N(t)$ is the constant N, while $M(t)$ is given by

$$M(t) = M_0 + 0.24(t-1)N, \tag{2.17}$$

where M_0 is the original total cash endowment that is supplemented by the dividend of 24 cents per share. If the dividend is based upon a random distribution, then the actual value can be used above.

Note that a maximum that typically occurs in a bubbles experiment satisfies (2.16) with $\mathrm{d}P/\mathrm{d}t$ set to zero, i.e. (2.11) with $L(t)$ redefined. This equation does not involve Q_1 directly, but since $L(t)$ and $P_\mathrm{a}(t)$ both vary in time, the influence of Q_1 appears to be through the time at which the maximum is attained. Given Q_2, the maximum value is determined by the time period.

This approach clarifies the role of the liquidity, by (at least in the continuum system) downplaying the role of the price derivative, since there is no delay. In the discretized version of (2.16), however, the time-steps representing periods effectively average the price derivative and provide for the delay or trend effect.

A few numerical studies of equations (2.16) and (2.17) help to illustrate some of the issues. In figure 1*a*–*d* one has a series of computations with all parameters identical, $Q_1 = 0$, $Q_2 = 0.2$, $P(0) = \$2.60$, except for the initial endowment of cash $M(0)$, which has the values \$2.40, \$3.00, \$3.60 and \$7.20, respectively. These can be regarded as liquidity-induced bubbles since Q_1 has been set at zero. The maximum value of price increases with increasing $M(0)$, and the peak appears to be very early when there is large cash endowment, such as the initial \$7.20 for each share of asset. The maximum value of nearly double (with this value of Q_2) the $P_\mathrm{a}(0)$ value when the cash endowment is twice the asset endowment is reminiscent of the single payout experiments (Caginalp *et al*. 1998), where the opening period featured prices that were close to the liquidity value of about twice the fundamental value under similar cash/asset endowment imbalances. Figure 2a, b displays the time evolution for $M(0) = 7.20$ and values of parameters that are identical to those of figure 1

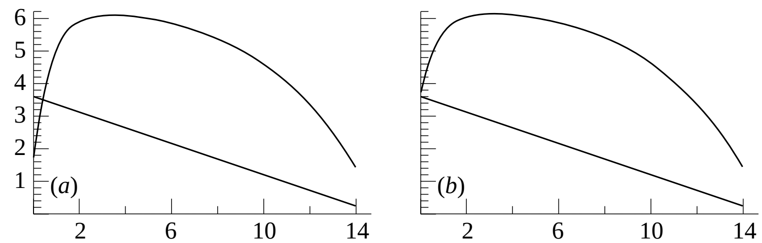

Figure 2.

except for initial price, with $P(0) = 1.60$ and $P(0) = 3.60$, respectively, so that the three initial price values of 1.60, 2.60 and 3.60 all result in an early bubble of about the same magnitude.

Although the numerical study of (2.16) involves a discretization into small time intervals, using the discretized version in which the time intervals are the periods in an asset market experiment essentially averages the price trend. This discrete equation is

$$\tau_0\left[1-\frac{Q_1}{P(t)}L(t)\right]\{P(t+1)-P(t)\} = \left[1+\frac{Q_2}{P_{\mathrm{a}}(t)}L(t)\right]P(t)+(1+Q_2)L(t), \quad (2.18)$$

and similar discrete equations can be written for the systems of equations in § 3.

Another interesting feature of the liquidity induced bubbles is the relative independence of the time and price at the maximum from the initial value. This is not surprising since one has $\mathrm{d}P/\mathrm{d}t = 0$ at the maximum so that (2.11) and (2.13) are valid (with the appropriate values of $L(t)$ at that time).

In addition, for experiments involving a single payout at the end, this feature is often important at the beginning of an experiment. Also, if the price evolves toward an asymptotic value it is valid near the end.

Note that in asset experiments it has been shown (Caginalp *et al*. 1999) that a lower initial price produces a larger bubble, due to the effect of the price trend, which is subordinated in this model, as will be seen when equation (2.16) is derived as a limit of the more general equations.

Thus it appears that the liquidity induced bubbles have a somewhat different character than those produced thus far in the laboratory (with a more balanced initial cash/asset ratio). Further experimentation will be needed to determine the relative importance of liquidity in comparison with momentum.

3. A system of equations with momentum and asset flow

We present a brief review of a system of ordinary differential equations that have been used to study a broad range of issues such as market crashes and the discount paradox in closed end funds, as well as experimental asset markets (see Caginalp & Balenovich 1996 and references therein). As in the simpler model presented above, the key ingredients involve the dependence of preference on the price derivative (as well as deviation from fundamental value) and the finiteness of traders' assets.

The relative price change at time t is given by $P(t)^{-1}\mathrm{d}P(t)/\mathrm{d}t$, so that the impact of this change at a later time t is given by this expression multiplied by $\mathrm{e}^{-c_1(t-\tau)}$, where c_1^{-1} is a measure of the 'memory length'. A sum of the impact of all previous price changes results in a mathematical expression for the trend-based component of the investor preference, which is given by

$$\zeta_1(t) \equiv q_1 c_1 \int_{-\infty}^{t} \mathrm{e}^{-c_1(t-\tau)} \frac{1}{P(\tau)} \frac{\mathrm{d}P(\tau)}{\mathrm{d}\tau}\, \mathrm{d}\tau, \tag{3.1}$$

where q_1 is an amplitude constant to be determined experimentally.

We assume the value-based investor's motivation to buy is proportional to the fractional discount, i.e. $(P_\mathrm{a}(t) - P(t))/P_\mathrm{a}(t)$. However, there is lag time in implementing this decision, which is a characteristic of the body of investors. That is, the longer this discount persists from actual value, the greater the number of investors who act upon it. The exponential function is again appropriate as a decay in the fraction of investors who have not yet acted. Thus, the value-based component ζ_2 may be written as

$$\zeta_2(t) \equiv q_2 c_2 \int_{-\infty}^{t} \mathrm{e}^{-c_1(t-\tau)} \left[\frac{P_\mathrm{a}(\tau) - P(\tau)}{P_\mathrm{a}(\tau)}\right] \mathrm{d}\tau, \tag{3.2}$$

where c_2^{-1} is the time-scale and q_2 is the amplitude of this term. A large value for c_2 means that investors take action very quickly when there is an over- or under-valuation.

The sum of the two terms ζ_1 and ζ_2 then results in the investor sentiment function

$$\zeta(t) = q_1 c_1 \int_{\infty}^{t} \mathrm{e}^{-c_1(t-\tau)} \frac{1}{P(\tau)} \frac{\mathrm{d}P(\tau)}{\mathrm{d}\tau}\, \mathrm{d}\tau + q_2 c_2 \int_{-\infty}^{t} \mathrm{e}^{-c_2(t-\tau)} \frac{P_\mathrm{a}(\tau) - P(\tau)}{P_\mathrm{a}(\tau)}\, \mathrm{d}\tau, \tag{3.3}$$

which expresses the tendency to buy (when positive) or sell (when negative). This function embodies the key assumption about investors, and will play a pivotal role in the rates of buying and selling. The price equation is given by adjustment to the excess demand, yielding

$$\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}t} = F\left(\frac{k}{1-k}\frac{1-B}{B}\right), \tag{3.4}$$

where F is an increasing function such that $F(1) = 0$, which is taken as

$$F(x) = \delta \log x,$$

where δ is a constant amplitude that scales time in (3.4). The finiteness of traders' assets means that B changes as the asset is bought and sold (first two terms below) and as the price changes (last term),

$$\frac{\mathrm{d}B}{\mathrm{d}t} = k(1-B) + (k-1)B + B(1-B)\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}t}, \tag{3.5}$$

$$k(t) := \tfrac{1}{2} + \tfrac{1}{2}\tanh(\zeta_1 + \zeta_2). \tag{3.6}$$

By differentiating ζ_1 and ζ_2 in (3.1) and (3.2) using the Leibnitz rule, one has

$$\frac{\mathrm{d}\zeta_1}{\mathrm{d}t} = c_1\left(\frac{q_1}{P}\frac{\mathrm{d}P}{\mathrm{d}t} - \zeta_1\right), \qquad \frac{\mathrm{d}\zeta_2}{\mathrm{d}t} = c_2\left(q_2\frac{P_\mathrm{a}(t) - P(t)}{P_\mathrm{a}(t)} - \zeta_2\right). \tag{3.7}$$

The system of equations (3.4)–(3.7) is then to be solved numerically with the parameters to be determined by closest fit with experiment as discussed in Caginalp & Balenovich (1994). One interesting limit of these equations is the 'long time limit' in which c_1 and c_2 approach zero while $F_1 = c_1 q_1$ and $F_2 = c_2 q_2$ are held fixed, yielding

$$\frac{\mathrm{d}\zeta_1}{\mathrm{d}t} = \frac{F_1}{P}\frac{\mathrm{d}P}{\mathrm{d}t}, \qquad \frac{\mathrm{d}\zeta_2}{\mathrm{d}t} = F_2 \frac{P_{\mathrm{a}}(t) - P(t)}{P_{\mathrm{a}}(t)}, \tag{3.8}$$

with the other equations unchanged.

This system has been used to make predictions on experiments using the following procedure. (a) One or more experiments are used to calibrate the F_1 and F_2 that provide the best fit. (b) Using only the initial trading price of an experiment, one can then make predictions for the price evolution for the entire experiment. In a more recent work (Caginalp *et al.* 1999), a period-by-period version of this model was used to predict a 'bubble' experiment. The model was improved in that the actual values of the price up to period N were used instead of the predicted prices in order to forecast the price at time $N + 1$. These out-of-sample predictions were compared with other forecasting methods including (i) random walk, (ii) other time-series (ARIMA) methods that involved price derivative, (iii) excess bids methods, and (iv) human forecasters.

The single equation model of § 2 can be derived from equations (3.4)–(3.7). However, before doing this, we consider a simplified version of these equations.

If we assume (2.3) and (2.5) as before and again use (2.7) for $k/(1-k)$, we obtain the following.

Closed market system with delay times. This is the *third-order ordinary differential equation* system given by

$$\frac{\tau_0}{P}\frac{\mathrm{d}P}{\mathrm{d}t} = (1 + 2\zeta_1 + 2\zeta_2)\frac{L(t)}{P} - 1 \tag{3.9}$$

coupled with the differential equations (3.7) for ζ_1 and ζ_2. Once again, any arbitrary flow of funds and shares can be incorporated into this equation through $M(t)$ and $N(t)$.

Note that the single equation and the system (3.7), (3.9) yield the same equilibrium results so that one again has the identities (2.11) and (2.13) for the equilibrium price as a function of the liquidity price and fundamental value.

A second-order system can be obtained by using (3.9) in conjunction with the differential equation for ζ_1 and the limit of small relaxation time $1/c_2$, namely,

$$\frac{\mathrm{d}\zeta_1}{\mathrm{d}t} = c_1\left(\frac{q_1}{P}\frac{\mathrm{d}P}{\mathrm{d}t} - \zeta_1\right), \qquad \zeta_2 = q_2\left(1 - \frac{P(t)}{P_{\mathrm{a}}(t)}\right). \tag{3.10}$$

Equations (3.9) and (3.10) describe a market that is influenced by price trend but also reacting to changes in fundamental value $P_{\mathrm{a}}(t)$ without delay.

We now derive the single (first-order) equation as a limit of equations (3.4)–(3.7) as c_1 and c_2 approach infinity within the closed market set-up. In this limit, equations (3.7) have the form

$$\zeta_1 = \frac{F_1}{P}\frac{\mathrm{d}P}{\mathrm{d}t}, \qquad \zeta_2 = F_2 \frac{P_{\mathrm{a}}(t) - P(t)}{P_{\mathrm{a}}(t)}, \tag{3.11}$$

which differs from the long time limit (3.8) in that the functions ζ_1 and ζ_2 rather than their time derivatives appear on the left-hand sides.

Next we begin with the definition of k in (3.6) and approximate the exponential function by the first term of the Taylor series, so that

$$\frac{k}{1-k} = \mathrm{e}^{2\zeta} \cong 1 + 2\zeta \tag{3.12}$$

and $k \cong \frac{1}{2}(1+\zeta)$. A small deviation of k from $\frac{1}{2}$ is adequate to provide a rapid price adjustment, so that this approximation, which is of order ζ^2, is not a very strong one.

Next we linearize the price equation (3.4), again using Taylor's theorem, yielding

$$F\left(\frac{D}{S}\right) - F(1) = F'(1)\left(\frac{D}{S} - 1\right) + O\left[\left(\frac{D}{S} - 1\right)^2\right], \tag{3.13}$$

so that the restrictions $F(0) = 0$ and $F'(0) > 0$ yield (2.3) with $\tau_0 := 1/F'(0)$.

Combining the relations above with the conservation law (2.5), which is just the internal, or last part of (3.5), one has (2.16). The short time limit that leads to (2.16) has the consequence of eliminating the delay in investor action. This eliminates the oscillations in the case, for example, when P_{a} = const., as one can see from the autonomous nature of these equations.

4. Stochastic differential equations

Equations (3.4)–(3.7) can be generalized to stochastic differential equations by adding a stochastic term to B in order to represent random changes. In terms of an aggregate market, this would represent the randomness associated with additional sums (or withdrawals) of cash coming into the market, as well as additional supply (or buy-outs) of shares. This is then a nonlinear stochastic equation. Alternatively, one could insert randomness into one of the equations, e.g. (3.5) in a linear manner. However, this makes the equations inconsistent and numerical studies have shown that these equations behave more erratically than the nonlinear version.

In these equations, then, we change B into $B + aW$, where W is a standard normal random variable and a is constant, with the same random term used throughout for that particular time iteration.

Equations (3.4)–(3.7) have been used without randomness (Caginalp & Balenovich 1996) to generate a number of patterns that are characteristic of market tops or consolidation. We use a fourth-order Runge–Kutta method with step size $\mathrm{d}t$ of 0.05 or 0.1 as indicated below. For example, if the fundamental value $P_{\mathrm{a}}(t)$ exhibits a rounded top (shown in dashed lines in figure 3*a*), indicating that the fortunes of the asset are gradually turning, the price $P(t)$ may exhibit oscillations about this value that are described as a head-and-shoulders pattern (shown in solid lines in figure 3*a*). The peak is accompanied by two lesser peaks on either side that are the result of oscillations due to the flow between cash and asset, as well as the derivative. The addition of randomness leads to a range of patterns as shown in figure 3*b*–*f*. The variety evident in these patterns is part of the difficulty in defining the patterns and the controversy about their predictive value.

In each of these computer runs, equations (3.4)–(3.7) were used with the function $P_{\mathrm{a}}(t)$ as shown in figure 3*a*, and initial conditions $B = 0.5$, $\zeta_1 = \zeta_2 = 0$. For figure 3*a*–*c*, the parameters were $c_1 = 0.001$, $c_2 = 0.01$, $q_1 = 900$ and $q_2 = 45$. The stochastic

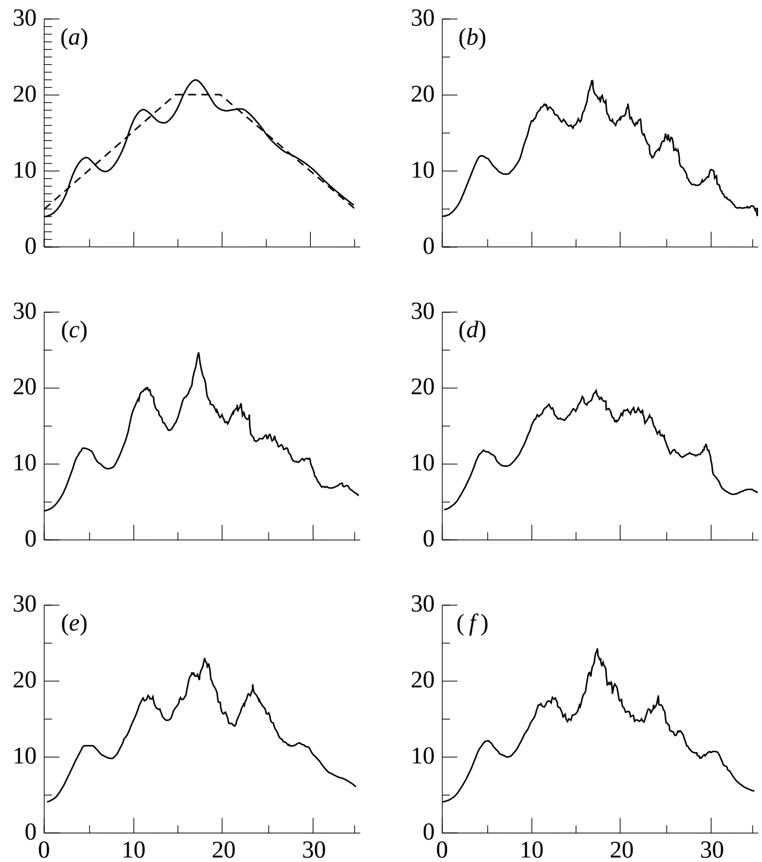

Figure 3.

values were a =0.001 75 and dt = 0.05 for figure 3b, c. For figure 3d, e, the parameters $c_1 = 0.01$, $c_2 = 0.001$, $q_1 = 90$ and $q_2 = 450$ were used, with stochastic values $a = 0.002$ and dt = 0.05 for figure 3d, $a = 0.002$ and dt = 0.1 for figure 3e, and $a = 0.0025$ and dt = 0.05 for figure 3f, along with parameters $c_1 = c_2 = 0.001$ and $q_1 = 900$, $q_2 = 450$.

Similarly, an inverted V-top for $P_{\rm a}(t)$ as shown in figure 4a leads to a $P(t)$ that is a smooth oscillation about the fundamental value. Figure 4b–f shows the effects of stochastic behaviour, as a sampling of the $P(t)$ evolutions are displayed. In each of these computer runs, equations (3.4)–(3.7) were used with parameters $c_1 = 0.001$, $c_2 = 0.001$, $q_1 = 875$ and $q_2 = 250$, the function $P_{\rm a}(t)$ has slopes ± 1 and reaches a maximum value of \$45, and initial conditions $B = 0.5$, $\zeta_1 = \zeta_2 = 0$ and $P(0) = 4$ were used. The stochastic values were a =0.0015 and dt = 0.05 for figure 4b, c, and $a = 0.002$ and dt = 0.1 for figure 4d.

In many market situations there is a sudden change in the value of an asset due to an unexpected event. If we assume a $P_{\rm a}(t)$ that exhibits an abrupt drop at some time, then the evolution of $P(t)$ will involve oscillations that overshoot the new fundamental value and gradually converge to it as shown in figure 5a. Figure 5b–d shows some of the patterns that arise from the stochastic behaviour. A feature that appears to be common to many of the patterns is the lack of much randomness

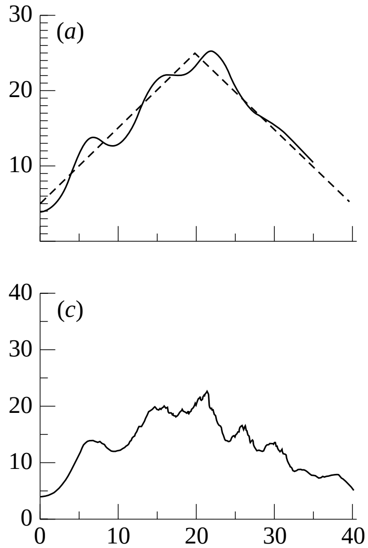

Figure 4.

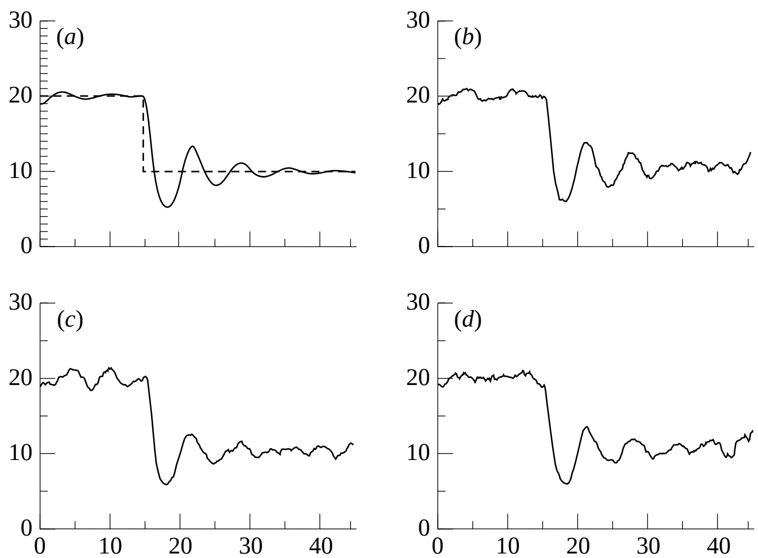

Figure 5.

during the rapid slide toward the bottoms. The randomness increases as the buying and selling is more balanced. In each of these computer runs, equations (3.4)–(3.7) were used with parameters $c_1 = 0.001$, $c_2 = 0.001$, $q_1 = 850$ and $q_2 = 450$, the function $P_a(t)$ exhibits a drop from 20 to 10, and we use initial conditions $B = 0.5$, $\zeta_1 = \zeta_2 = 0$ and $P(0) = 19$, The stochastic values were $a = 0.006$ and $\mathrm{d}t = 0.05$ for figure 5*b*, *c*, and $a = 0.0065$ and $\mathrm{d}t = 0.05$ for figure 5*d*.

For a basic trend with a stepwise declining $P_a(t)$ as shown in figure 6*a*, one has a gradually oscillating $P(t)$ when there is no randomness. The addition of stochastics generates a set of trend-lines with random fluctuations (figure 6*b*–*d*). In each of these

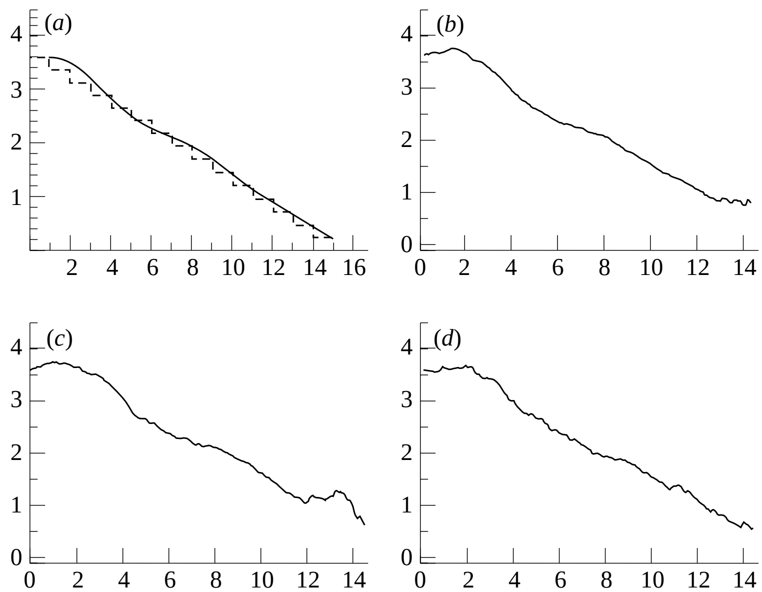

Figure 6.

computer runs, equations (3.4)–(3.7) were used with parameters $c_1 = 0.052$, $c_2 = 1$, $q_1 = 23.45$ and $q_2 = 0.06$, the function $P_a(t) = 3.84 - 0.24t$, and initial conditions $B = 0.5$, $\zeta_1 = \zeta_2 = 0$ and $P(0) = 3.6$. The stochastic values were $a = 0.008$ and $dt = 0.1$ for figure 6*b*, and $a = 0.015$ and $dt = 0.05$ for figure 6*c*, *d*.

These numerical studies indicate that the market trends and tops described by the stochastic equations exhibit a great deal of diversity that is characteristic of markets.

5. Conclusion

We have derived a new and simple model that demonstrates the importance of liquidity in markets. In particular, the price and fundamental value of an asset are complemented by a liquidity value $L(t)$, which is the total cash divided by the total number of shares at time t. Analytical and numerical results predict that a higher cash supply will result in a larger bubble.

This analysis provides a mathematical framework that helps justify the belief that 'cheap money' or high liquidity in a market is a major factor that moves asset prices higher. An accomodative stance by a central bank is a key factor in this process.

In many cases, numerous sources of high liquidity are coincident, as in the US markets in the 1990s, as large amounts of cash have flowed into the stock markets due to tax policies, easy credit, low interest rates that were partly in response to a banking crisis, the baby boom generation becoming older and investing more, etc. As prices of many stocks have lost connection with traditional value measures, it is clear that the liquidity has been an important factor in stock prices.

The number of available shares of particular types of companies is also a significant factor in the liquidity picture. In times of rapid takeovers that outstrip the pace of new and secondary issues, the shrinking supply of stock also creates a high-liquidity environment for rising prices. This has been described as one of the reasons for the bull market of the 1980s.

Advantages of our simple model include the relatively modest assumptions made in its derivation and the fact that the key conclusions on the importance of liquidity are not dependent on supply or demand's dependence on the price derivative. In particular, the only economic assumption is the price change due to excess demand, and the dependence of supply/demand on fundamental value and price derivative. The mathematically simplifying assumption of linearity (that is usually present in price theory) does not play a significant role in the conclusions. The basic equations such as (2.4) and (2.5) are mathematical identities of a closed system. Alternatively, the equation can be derived as a limit of a larger system of equations involving delay.

The single-equation model with variable cash and stock also provides a framework for understanding in a simple way a spectrum of national crises such as the drain of foreign investment, speculator attack, etc. One can also simulate these within a laboratory setting to confirm the results.

For example, in a single pay-out experiment one can begin with equal total endowments of cash and asset and then introduce additional players who have a surplus of cash during the middle periods. This should produce a bubble due to the liquidity. Similarly, the introduction of players with a surplus of shares would tend to lower prices below the fundamental value.

Another prediction on the standard bubble experiments would be that an experiment which deferred dividends to the end of the experiment would have a smaller bubble; and one which distributed them as stock would have a still smaller bubble.

Equations (3.7), (3.9) offer the full price trend effect within the closed-market system at the expense of a somewhat more complicated set of equations. Both the single equation and the system (3.7), (3.9) are ideal for a precise study of experimental asset markets since one can take into account the dividends that are distributed exactly. Since one can vary the number of shares and the amount of cash arbitrarily, these equations can be used to model a broad spectrum of experiments that can be designed.

The stochastics of the last section show that a diverse set of consolidation or peaking patterns are obtained from the larger system of equations when randomness is included. Despite the small number of parameters involved in the equations, the computer studies have shown a rich structure illustrated by the examples of figures 3–6.

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